A CONNECTION BETWEEN PLÜCKER COORDINATES AND POLYNOMIAL RINGS

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Abstract. Given a polynomial ring R with finitely many indeterminants over a field, we will investigate a technique useful in predicting whether the vector space spanned by the dth degree monomials has image zero under the canonical projection $\varphi : R \to R/I$ when I is an ideal of R generated by linearly independent homogeneous polynomials of like degree lower than d.

Introduction.

In this paper, we explore a relationship [1] between finite dimensional commutative algebras (i.e. quotients of polynomial rings) and projective varieties. In particular, we will show that that if I is an ideal of $k[x_1, \ldots, x_n]$ generated by m linearly independent d_1th degree polynomials whose coefficient matrix is given by A, then there exists a projective variety V such that the vector space of homogeneous d_2th degree polynomials has image zero under the canonical projection $\varphi : R \to R/I$ (see definition 1.9 below) if and only if $\Lambda_m(A)^T \notin V$.

1. SOME DEFINITIONS AND RESULTS.

To better understand the results contained herein, it is necessary to familiarize oneself with the following definitions and theorems, many of which are from linear algebra. Unless otherwise stated, k is an algebraically closed field.

1.1. Let A be an $m \times n$ matrix over k. Then A represents a linear transformation $T \in L(k^n, k^m)$. Let j be a positive integer such that $j \leq \min\{m, n\}$. Then the matrix $\Lambda_j(A)$ representing $\Lambda_j(T)$, the jth exterior power of T, consists of $j \times j$ minors of A, and $\Lambda_j(A)$ is an $\begin{pmatrix} m \\ j \end{pmatrix} \times \begin{pmatrix} n \\ j \end{pmatrix}$ matrix over k. [2]

1.2. Let A be an $n \times n$ matrix over k. Then $\Lambda_n(A) = \det(A)$.

1.3. Let $m, n \in \boldsymbol{Z}^+$ such that $m \leq n$. Let A be an $n \times m$ matrix over k. Let $P = \Lambda_m(A)^T$, a row vector with $t = \begin{pmatrix} n \\ m \end{pmatrix}$ entries. Then $P = (P_1, \ldots, P_t)$, where the P_i are called Plücker coordinates, represents a point in the projective space $\boldsymbol{P}^{t-1}$ with respect to A. And if B is another $n \times m$ matrix over k such that

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[1]This problem is motivated by a search for possible classification schemes of certain Gorenstein rings; in particular, we wished to classify all Gorenstein rings whose Hilbert series is $H(t) = 1 + 5t + 5t^2 + t^3$. A Gorenstein ring which possessed this Hilbert series provided a counterexample to one of Auslander's conjectures. For a treatise on Gorenstein rings, see Huneke. For the counterexample to Auslander's claim, see Jorgensen and Sega.

[2]See Curtis (256-258) for a brief introduction of the exterior product.

the column spaces of A and B are the same, then $\Lambda_m(A)^T = \Lambda_m(B)^T \in \boldsymbol{P}^{t-1}$. Therefore, we obtain a well-defined map from the set of m-dimensional subspaces of k^n to the projective space $\boldsymbol{P}^{t-1}$. [3]

1.4. **Example.**
Let $k = \boldsymbol{R}$, $m = 2$, $n = 3$. Let

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 1 \end{pmatrix}.$$

So

$$\Lambda_2(A) = (2 \cdot 0 - 1 \cdot 3, 2 \cdot 1 - 1 \cdot 1, 3 \cdot 1 - 0 \cdot 1) = (-3, 1, 3). \tag{1.4.1}$$

So the Plücker point with respect to A is (-3, 1, 3).

1.5. Let A be an $n \times m$ matrix over k, $n \leq m$. Let $x \in k^m$, $b \in k^n$, such that $Ax = b$ and suppose A has maximum rank. Then $x = \mathbf{0}$ if and only if $b = \mathbf{0}$.

1.6. Suppose $R = k[x_1, \ldots, x_n]$, d is a positive integer. Then the vector space dimension of the space spanned by the dth degree monomials is equal to the number of dth degree monomials, and that number is $w_d = \begin{pmatrix} n+d-1 \\ d \end{pmatrix}$. We will use the notation w_d throughout the paper. (Though it seems that this notation will prove to be ambiguous, the context in which the symbol appears will elucidate the meaning.)

1.7. Let $f_1, \ldots, f_m$ be homogeneous dth degree polynomials in $k[x_1, \ldots, x_n]$. Then, for the purposes of this paper, $f_1, \ldots, f_m$ are said to be linearly independent if they are linearly independent as vector space elements. That is, if $a_i \in k$ where $i = 1, \ldots, m$, then $a_1 f_1 + \cdots + a_m f_m = 0$ if and only if $a_i = 0$ for $i = 1, \ldots, m$.

1.8. Let $R = k[x_1, \ldots, x_n]$, d be a positive integer. Since the vector space of dth degree homogeneous polynomials is spanned by the w_d dth degree monomials $x_1^d, x_1^{d-1}x_2, \ldots, x_n^d$, a homogeneous dth degree polynomial f in R can be written as

$$f = a_1 x_1^d + a_2 x_1^{d-1} x2 + \cdots + a_{w_d} x_n^d, \tag{1.8.1}$$

where $a_i \in k$ for $i = 1, \ldots, w_d$. We will use the lexicographic order in listing the basis monomials.

1.9. Let A be an $m \times n$ matrix over k. Let j be a positive integer such that $j \leq \min\{m, n\}$. Then $\operatorname{rank}(A) = j$, a positive integer, if and only if at least one $j \times j$ minor of A is nonzero and all minors of dimension greater than j in A are zero.

1.10. Let R be a commutative ring, I an ideal of R. Then there exists a ring homomorphism $\varphi : R \to R/I$ such that $\varphi(r) = r + I$ for all $r \in R$. We call φ the canonical projection of R onto R/I.

[3] See Hodge and Pedoe for a more detailed discussion of Plücker coordinates.

2. A PRIMER FOR PLÜCKERS

In considering an approach to the problem of classification, let us explore the following problem:

2.1. Let I be an ideal of $R = k[x, y]$ such that I is generated by two linearly independent homogeneous quadratic polynomials f_1 and f_2. We will exact a method by which we may determine whether the vector space of homogeneous cubics has image zero under $\varphi : R \to R/I$.

Since the vector space of homogeneous quadratic polynomials is spanned by x^2, xy, and y^2, two homogeneous quadratics may be written as

$$f_1 = a_{11}x^2 + a_{21}xy + a_{31}y^2 \tag{2.1.1}$$

and

$$f_2 = a_{12}x^2 + a_{22}xy + a_{32}y^2, \tag{2.1.2}$$

where $a_{ij} \in k$ for all i, j. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix},$$

the matrix of coefficients of the two generators of I. Since the two generators f_1 and f_2 are linearly independent, $\text{rank}(A) = 2$. By 1.9, at least one 2×2 minor of A is nonzero.

2.2. Now, recall from 1.3, the Plücker point with respect to A is

$$P = \Lambda_2(A)^T = (a_{11}a_{22} - a_{21}a_{12}, a_{11}a_{32} - a_{12}a_{31}, a_{21}a_{32} - a_{22}a_{31}). \tag{2.2.1}$$

Next, let us determine the subspace of the space of homogeneous cubics generated by the two generators of I.

In $k[x, y]$, to determine the homogeneous cubic polynomials generated by homogeneous quadratic polynomials, we need only consider the span of the homogeneous cubics obtained by shifting each of the given quadratics f_1 and f_2 by the homogeneous linear polynomials x and y. Since we will shift f_1 and f_2 each by x and y, we will obtain a generating set consisting of four homogeneous cubics. We may summarize the results of said shift in the following way:

Let $(X|Y) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$.

If we multiply the matrix

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

representing the homogeneous quadratic $f = ax^2 + bxy + cy^2$ by the first three columns of $(X|Y)$, we will obtain

$$\begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix},$$

a matrix providing the coefficients of the homogeneous cubic $ax^3+bx^2y+cxy^2+0y^3$ obtained by shifting f by x. If we multiply

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

by the last three columns of $(X|Y)$, we obtain

$$\begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix},$$

a matrix providing the coefficients of the homogeneous cubic $0x^3+ax^2y+bxy^2+cy^3$ obtained by shifting f by y.

So if we multiply $(X|Y)$ by $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \oplus \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, we obtain a 4×2 matrix providing the coefficients of the homogeneous cubic generators of the vector space generated by sending f into the homogeneous cubics. (This vector space, of course, is a subspace of the vector space of homogeneous cubics in $k[x,y]$.) The two homogeneous cubics obtained from the product matrix are the generators of the subspace of homogeneous cubics which has image zero under $\varphi : R \to R/I$.

2.3. So let us apply this matrix multiplication in our problem:

Let $(X|Y)(A\oplus A) = B$. Thus, B is a 4×4 matrix. The multiplication and the results are as follows:

(2.3.1)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \cdot \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \\ 0 & 0 & a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{11} & a_{12} \\ a_{31} & a_{32} & a_{21} & a_{22} \\ 0 & 0 & a_{31} & a_{32} \end{pmatrix}.$$

Now, since the dimension of the vector space of homogeneous cubic polynomials in $k[x,y]$ is 4, we need $\operatorname{rank}(B) = 4$ if f_1 and f_2 are to generate all homogeneous cubics in $k[x,y]$. Thus, the vector space of homogeneous cubics has image zero under $\varphi : R \to R/I$ if and only if $\operatorname{rank}(B) = 4$.

But since B is a 4×4 matrix, $\operatorname{rank}(B) = 4$ if and only if $\det(B) \neq 0$. So the problem is reduced to the evaluation of a single determinant. Now, cofactor expansion reveals that

$$\det(B) = a_{11}(a_{22}P_3 - a_{32}P_2) - a_{21}a_{12}P_3 + a_{31}a_{12}P_2, \tag{2.3.2}$$

and we may use a little algebra to show that

$$\det(B) = P_1P_3 - P_2^2. \tag{2.3.3}$$

Thus, f_1 and f_2 generate all homogeneous cubics in $k[x,y]$ if and only if $P_1P_3 - P_2^2 \neq 0$. Alas, we have a result:

2.4. **Theorem.** *Let $R = k[x, y]$, I be an ideal of R generated by two linearly independent homogeneous quadratics f_1 and f_2, A be the matrix of coefficients of f_1 and f_2 with respect to the monomial basis $\{x^2, xy, y^2\}$, and $(P_1, P_2, P_3) = \Lambda_2(A)^T$. Then the vector space of homogeneous cubics has image zero under $\varphi : R \to R/I$ if and only if the Plücker quantity $P_1P_3 - P_2^2$ is nonzero.*

Now, let us consider an application of this powerful result:

2.5. **Example** Suppose $R = k[x, y]$, $I = (x^2 + xy + y^2, 2x^2 - xy)$. Then the vector space of homogeneous cubic polynomials has image zero under $\varphi : R \to R/I$.

Proof. Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$ which gives the coefficients of the two homogeneous quadratics listed above. Let $(P_1, P_2, P_3) = \Lambda_2(A)^T = (-3, 2, -1)$. Since at least one coordinate is nonzero (i.e. a 2×2 minor of A is nonzero), we have that the two homogeneous quadratics are linearly independent. Since $P_1P_3 - P_2^2 = 3 - 4 = -1$ is nonzero, theorem 2.4 tells us that the vector space of homogeneous cubics has image zero under $\varphi : R \to R/I$. □

This process suggests that the Plücker quantities may, in fact, govern entirely whether a set of homogeneous dth degree polynomials generate the vector space of homogeneous $(d + 1)$st degree polynomials in $k[x_1, \ldots, x_n]$.

3. THE NEXT STEP

In attempting to generalize our result, we cannot apply 2.4 since the proof relies on the fact that B is a square matrix. So let us investigate the following problem, a problem in which the matrix we earlier named B is not square:

3.1. Let $R = k[x, y]$ and I be an ideal of R generated by three linearly independent homogeneous cubic polynomials f_1, f_2, and f_3. Again, we will explore conditions under which we may be assured that the vector space spanned by homogeneous quartic polynomials has image zero under $\varphi : R \to R/I$.

Since f_1, f_2, and f_3 are three linearly independent homogeneous cubics in R, we may write

$$f_j = a_{1j}x^3 + a_{2j}x^2y + a_{3j}xy^2 + a_{4j}y^3 \tag{3.1.1}$$

for $j = 1, 2, 3$, where $a_{ij} \in k$ for all i, j. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$ be the matrix of coefficients of f_1, f_2, and f_3. Let $P = (P_1, P_2, P_3, P_4) = \Lambda_3(A)^T$. In this case, the matrix

$$(X|Y) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{3.1.2}$$

As before, let $B = (X|Y)(A \oplus A)$. So we obtain

$$B = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}. \tag{3.1.3}$$

Now, the vector space of homogeneous quartic polynomials has image zero under $\varphi : R \to R/I$ if and only if $\operatorname{rank}(B) = 5$. Thus, the vector space of homogeneous quartics has image zero under $\varphi : R \to R/I$ if and only if at least one 5×5 minor of B is nonzero. But since B is not a square matrix, we must evaluate multiple determinants to assure that B is full rank. And, in general, these determinants will not be Plücker quantities.

3.2. However, observe that if we could somehow enlarge B to become a square matrix, it is possible that we could obtain possibly more than one Plücker quantity determining whether the space of homogeneous quartics has image zero under $\varphi : R \to R/I$. And, as it turns out, we can accomplish this as follows:

Since B is a 5×6 matrix, we may append an additional row to the top of B to obtain a 6×6 matrix. So let us append rows taken from $A \oplus A$. Since there are eight rows in $A \oplus A$, we may obtain eight different 6×6 matrices by appending rows to the top of B. Let B_j be the 6×6 matrix obtained by appending the jth row of $A \oplus A$ to the top of B.

So here are the eight matrices:

$$B_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}, B_2 = \begin{pmatrix} a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}$$

$$B_3 = \begin{pmatrix} a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}, B_4 = \begin{pmatrix} a_{41} & a_{42} & a_{43} & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}$$

$$B_5 = \begin{pmatrix} 0 & 0 & 0 & a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}, B_6 = \begin{pmatrix} 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}$$

$$B_7 = \begin{pmatrix} 0 & 0 & 0 & a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}, B_8 = \begin{pmatrix} 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \\ a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}.$$

3.3. Using cofactor expansion, we may show that the determinant of each of these matrices is a linear combination of the elements on the first row and the 5×5 minors of B critical to our problem. So let us assign these six 5×5 minors of B to a column vector to help us evaluate these determinants. However, unlike in the case of using the exterior power, let us include in each minor the cofactor sign necessary in evaluating the determinants of the $B_j s$.

Let $M = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{pmatrix}$, where

$$M_1 = \det \begin{pmatrix} a_{12} & a_{13} & 0 & 0 & 0 \\ a_{22} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{32} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{42} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}, M_2 = -\det \begin{pmatrix} a_{11} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{23} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{33} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{43} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}$$

$$M_3 = \det \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{31} & a_{32} & a_{33} \\ 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix}, M_4 = -\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{42} & a_{43} \end{pmatrix}$$

$$M_5 = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{23} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{43} \end{pmatrix}, M_6 = -\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{11} & a_{12} \\ a_{31} & a_{32} & a_{33} & a_{21} & a_{22} \\ a_{41} & a_{42} & a_{43} & a_{31} & a_{32} \\ 0 & 0 & 0 & a_{41} & a_{42} \end{pmatrix}$$

Now, both B_1 and B_8 are singular matrices regardless of the values of the possibly nonzero entries, so

$$\det(B_1) = a_{11}M_1 + a_{12}M_2 + a_{13}M_3 = 0 \tag{3.3.1}$$

and

$$\det(B_8) = a_{41}M_4 + a_{42}M_5 + a_{43}M_6 = 0. \tag{3.3.2}$$

So of particular interest is a case such as that of B_2, a possibly nonsingular matrix.

Now, notice that

$$\det(B_2) = a_{21}M_1 + a_{22}M_2 + a_{23}M_3. \tag{3.3.3}$$

With a bit of algebra, it can be shown that $\det(B_2) = P_1P_3 - P_2^2$.
Thus, the determinant of B_2 is a Plücker quantity.
Similarly,

$$\det(B_3) = a_{31}M_1 + a_{32}M_2 + a_{33}M_3 = P_1P_4 - P_2P_3 \tag{3.3.4}$$

and

$$\det(B_4) = a_{41}M_1 + a_{42}M_2 + a_{43}M_3 = P_2P_4 + P_3^2. \tag{3.3.5}$$

Now, consider B_5: if we interchange rows one and two, subtract row three from row one, and finally add row one to row three, we obtain B_2. Thus, we have that $\det(B_5) = -\det(B_2)$. So

$$\det(B_5) = a_{11}M_4 + a_{12}M_5 + a_{13}M_6 = P_2^2 - P_1P_3. \tag{3.3.6}$$

Using a similar strategy, it can be shown that $\det(B6) = -\det(B3)$. Moreover, $\det(B7) = -\det(B4)$. So

$$\det(B6) = a_{21}M_4 + a_{22}M_5 + a_{23}M_6 = P_2P_3 - P_1P_4 \tag{3.3.7}$$

and

$$\det(B7) = a_{31}M_4 + a_{32}M_5 + a_{33}M_6 = -P_2P_4 - P_3^2. \tag{3.3.8}$$

As before, the relevant determinants equal either Plücker quantities or zero. So the question comes to mind, how can we relate this to the result in 2.4?
First, notice that equations 3.3.1-3.3.8 give a system of linear equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 & a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 & a_{41} & a_{42} & a_{43} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{pmatrix} = \begin{pmatrix} 0 \\ P_1P_3 - P_2^2 \\ P_1P_4 - P_2P_3 \\ P_2P_4 + P_3^2 \\ -P_1P_3 - P_2^2 \\ -P_1P_4 - P_2P_3 \\ -P_2P_4 + P_3^2 \\ 0 \end{pmatrix}. \tag{3.3.9}$$

Letting J be the right hand side of the equation, we may write

$$(A \oplus A)M = J. \tag{3.3.10}$$

Now, since $\operatorname{rank}(A) = 3$, $\operatorname{rank}(A \oplus A) = 6$. Since $A \oplus A$ is of maximum rank, by 1.5, all $M_i = 0$ if and only if every entry of J is zero. And if every 5×5 minor M_i of B is zero, the homogeneous cubics f_1, f_2, and f_3 do not generate all the homogeneous quartics.

Thus, we have the following:

3.4. **Theorem.** *Let $R = k[x, y]$, I be an ideal of R generated by three linearly independent homogeneous cubic polynomials f_1, f_2, f_3, A be the coefficient matrix of f_1, f_2, and f_3 with respect to the monomial basis $\{x^3, x^2y, xy^2, y^3\}$, and $(P_1, P_2, P_3, P_4) = \Lambda_3(A)^T$. Then the vector space of homogeneous quartic polynomials has image zero under $\varphi : R \to R/I$ if and only if at least one of the three Plücker quantities $P_2^2 - P_1P_3$, $P_3^2 + P_2P_4$, and $P_1P_4 - P_2P_3$ is nonzero.*

We may obtain a more explicit description of the projective variety defined by the three equations $P_2^2 - P_1P_3 = 0$, $P_3^2 + P_2P_4 = 0$, and $P_1P_4 - P_2P_3 = 0$. Under the hypotheses of theorem 3.4, at least one Plücker coordinate is nonzero. So we may analyze this problem in four cases:

Case I: Suppose $P_1 = \alpha$ nonzero. Since $P_1 = \alpha$, we have

$$P_2^2 = \alpha P_3, \tag{3.4.1}$$

$$P_3^2 = -P_2P_4, \tag{3.4.2}$$

and

$$\alpha P_4 = P_2P_3. \tag{3.4.3}$$

Solving for P_3 in equation 3.4.1 and substituting into equations 3.4.2 and 3.4.3 gives

$$P_2^4/\alpha^2 = -P_2P_4, \tag{3.4.4}$$

and

$$\alpha P_4 = P_2^3/\alpha. \tag{3.4.5}$$

By solving for P_4 in equation 3.4.5 and substituting into equation 3.4.4 we obtain

$$P_2^4/\alpha^2 = -P_2^4/\alpha^2, \tag{3.4.6}$$

which simplifies to

$$2P_2^4 = 0. \tag{3.4.7}$$

Since k, by hypothesis, is infinite, we have that 2 is nonzero. Thus, by equation 3.4.7, $P_2 = 0$. Since $P_2 = 0$ and $P_2^2 - P_1P_3 = 0$, we have that $P_3 = 0$. Since $P_3 = 0$ and $P_1P_4 - P_2P_3 = 0$, we have that $P_4 = 0$.

Case II: A similar argument shows that if P_4 is nonzero, then $P_1 = P_2 = P_3 = 0$.

Case III: Suppose P_2 is nonzero. Thus, $P_2^2 - P_1P_3 = 0$ implies that both P_1 and P_3 are both nonzero. But by Case 1, P_1 being nonzero implies that $P_2 = 0$, a contradiction. Thus, $P_2 = 0$.

Case IV: We need not suppose that P_3 is nonzero since $P_3^2 + P_2P_4 = 0$ and $P_2 = 0$ imply that $P_3 = 0$.

Summarizing, if our three Plücker quantities are zero, then the second and third Plücker coordinates, as well as exactly one of the first or fourth Plücker coordinates, taken from the coefficient matrix A are zero.

So consider the following corollary to 3.4, a more streamlined result:

3.5. **Corollary.** *Let $R = k[x, y]$, I be an ideal of R generated by three linearly independent homogeneous cubic polynomials f_1, f_2, f_3, A be the coefficient matrix of f_1, f_2, and f_3 with respect to the monomial basis $\{x^3, x^2y, xy^2, y^3\}$, and $P = (P_1, P_2, P_3, P_4) = \Lambda_3(A)^T$. Then the vector space of homogeneous quartic polynomials does not have image zero under $\varphi : R \to R/I$ if and only if $P \in \{(1, 0, 0, 0), (0, 0, 0, 1)\}$, a projective variety in $\boldsymbol{P}_3$.*

Now, theorem 3.4 strongly suggests that the result can be generalized to polynomial rings of n indeterminants. And, in fact, we can perform such a generalization.

But before turning out attention to the final result of this paper, consider the following example:

3.6. **Example** Let $R = k[x, y]$. Let I be generated by $f_1 = x^3 - x^2y + y^3$, $f_2 = 2x^2y+3xy^2+3y^3$, and $f_3 = x^3+4xy^2+5y^3$. Then the vector space of homogeneous quartic polynomials has image zero under $\varphi : R \to R/I$.

Proof. Notice, the coefficient matrix A with respect to the monomial basis is $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 3 & 4 \\ 1 & 3 & 5 \end{pmatrix}$.

Now, let $P = (P_1, P_2, P_3, P_4) = \Lambda_3(A)^T$. So $P_1 = 5$, $P_2 = 5$, $P_3 = 0$, $P_4 = 5$. Since at least one Plücker coordinate is nonzero, we know that $\text{rank}(A) = 3$, and thus f_1, f_2, and f_3 are linearly independent. Thus, we may use 3.5. Since $(5, 5, 0, 5)$ is not a point on the projective variety described in 3.5, we have that the space of homogeneous quartics has image zero under $\phi : R \to R/I$. □

4. THE GENERAL RESULT

Though theorems 2.4 and 3.4 provide conditions under which an ideal generated by $w_d - 1$ homogeneous dth degree polynomials generate all homogeneous $(d + 1)$st degree polynomials in two specific cases, it is possible to generalize not only the number of indeterminants of R, but also the jump from a starting space of homogeneous d_1th degree polynomials to the space of homogeneous d_2th degree polynomials in R. And the result to follow provides conditions under which we may accomplish this. But to show it, we need a lemma:

4.1. **Lemma.** *Let A be an $n \times m$ matrix over k where $m \leq n$, and j be a positive integer such that $j \leq m$. Then* $\text{rank}(A) = m$ *if and only if* $\text{rank}(\Lambda_j(A)) = \begin{pmatrix} m \\ j \end{pmatrix}$.

Proof. Note that A represents a linear transformation $T : k^m \to k^n$. Suppose $\text{rank}(A) < m$. Then there exists a basis $\{v_1, \dots, v_m\}$ of k^m such that $T(v_1) = 0$. From $\{v_1, \dots, v_m\}$, we may generate a basis of $\Lambda_j(k^m)$ using the standard ordering. Thus $\{v_{i_1} \wedge \cdots \wedge v_{i_j} | i_1 < \cdots < i_j, i_t = 1 \text{ to } m \text{ for } t = 1 \text{ to } j\}$ is a basis of $\Lambda_j(k^m)$. And $\Lambda_j T$ is a linear transformation from $\Lambda_j k^m$ to $\Lambda_j k^n$, defined by $\Lambda_j T(u_1 \wedge \cdots \wedge u_j) = T(u_1) \wedge \cdots \wedge T(u_j)$, for $u_i \in k^m$. So $\Lambda_j T(v_1 \wedge v_2 \wedge \cdots \wedge v_j) = T(v_1) \wedge \cdots \wedge T(v_j) = 0 \wedge \cdots \wedge T(v_j) = 0$. Thus, $\Lambda_j T$ is not injective. Thus, the matrix representing $\Lambda_j T$, $\Lambda_j(A)$, has rank less than $\begin{pmatrix} m \\ j \end{pmatrix}$. Now, suppose rank$(A) = m$. Let $\{v_1, \ldots, v_m\}$ be a basis of k^m. Since A is of full rank, T is injective. Thus, $L = \{T(v_1), \ldots, T(v_m)\}$ is linearly independent. Furthermore, L is at least part of a basis of k^n. So let $T(v_i) = u_i$ for $i = 1$ to m. Again, $\{v_{i_1} \wedge \cdots \wedge v_{i_j} | i_1 < \cdots < i_j, i_t = 1 \text{ to } m \text{ for } t = 1 \text{ to } j\}$ is a basis of $\Lambda_j(k^m)$. And $\Lambda_j T(v_{i_1} \wedge \cdots \wedge v_{i_j}) = T(v_{i_1}) \wedge \cdots \wedge T(v_{i_j}) = u_{i_1} \wedge \cdots \wedge u_{i_j}$, for all such basis vectors of $\Lambda_j(k^m)$. But the left-hand side of the equation gives a basis vector of $\Lambda_j(k^n)$. Thus, the vectors obtained are part of a basis of $\Lambda_j(k^n)$, and they are linearly independent. Thus, $\Lambda_j T$ is injective. Thus, $\Lambda_j(A)$ is full rank. □

4.2. **Theorem.** *Let $R = k[x_1, \ldots, x_n]$, m, d_1, d_2 be positive integers such that $m < w_{d_1}$, $d_1 < d_2$,$mw_{d_2-d_1} \geq w_{d_2}$, and I be an ideal of R generated by m linearly independent homogeneous d_1th degree polynomials. Furthermore, let $h = \begin{pmatrix} w_{d_1} \\ m \end{pmatrix}$, A be the coefficient matrix with respect to the standard basis (in lexicographic order) of the aforementioned m homogeneous d_1th degree polynomials, and $P = (P_1, \ldots, P_h) = \Lambda_m(A)^T$. Then there exist homogeneous polynomial functions, some of which could be the zero function, $g_1, \ldots, g_s \in k[\xi_1, \ldots, \xi_h]$, of degree $w_{d_2-d_1}$, such that the vector space of homogeneous d_2th degree polynomials has image zero under $\varphi : R \to R/I$ if and only if at least one $g_1(P), \ldots, g_s(P)$ is nonzero.*

Proof. As before, to obtain the generators of the subspace of the vector space of homogeneous d_2th degree polynomials generated by the m linearly independent homogeneous d_1th degree polynomials, we must shift those m polynomials by the monomials of degree $d_2 - d_1$.

Thus, we must create a shift matrix $X = (y_1 | \ldots | y_{w_{d_2-d_1}})$, where (y_i) is a matrix that will shift the coefficients of a homogeneous d_1th degree polynomial f by the ith monomial of degree $d_2 - d_1$ to the coefficients of the homogeneous d_2th degree polynomial obtained by multiplying f by y_i. To shift all of the m homogeneous d_1th degree polynomials, we multiply X by $\oplus_{i=1}^{w_{d_2-d_1}} A$. Let $B = X(\oplus_{i=1}^{w_{d_2-d_1}} A)$.

In general, B, a $w_{d_2} \times mw_{d_2-d_1}$ matrix, is not square. However, by hypothesis, $mw_{d_2-d_1} \geq w_{d_2}$. Thus, B is at least as wide as it is tall. Now, the m homogeneous d_1th degree polynomials generate all of the homogeneous d_2th degree polynomials in R if and only if at least one $w_{d_2} \times w_{d_2}$ minor of B is nonzero.

As before, let us add rows to B to obtain a square matrix. Since B requires $z = mw_{d_2-d_1} - w_{d_2}$ additional rows to be square, let us add z rows, taken from $\oplus_{i=1}^{w_{d_2-d_1}} A$, to the top of B to create $t = \begin{pmatrix} w_{d_1} w_{d_2-d_1} \\ z \end{pmatrix}$ different $mw_{d_2-d_1} \times mw_{d_2-d_1}$ matrices. Call them $B_1, \ldots, B_t$, ordered as we would order the rows of $\Lambda_z(\oplus_{i=1}^{w_{d_2-d_1}} A)$. (That is to say, if the jth row of $\Lambda_z(\oplus_{i=1}^{w_{d_2-d_1}} A)$ is obtained by using the rows r_{j_1}th, $\ldots$, r_{j_z}th of $\oplus_{i=1}^{w_{d_2-d_1}} A$, then $B_j = \begin{pmatrix} r_{j_1} \\ \vdots \\ r_{j_z} \\ \hline \\ B \end{pmatrix}$. And if $z = 0$, then let $\Lambda_z(\oplus_{i=1}^{w_{d_2-d_1}} A) = |1|$, and let $B_1 = B$.)

Now, let J be a column vector whose entries are $\{\det(Bj)|j = 1 \text{ to } t\}$. Let $q = \begin{pmatrix} mw_{d_2-d_1} \\ w_{d_2} \end{pmatrix}$, the number of $w_{d_2} \times w_{d_2}$ minors of B. Let M be a column vector of those $w_{d_2} \times w_{d_2}$ minors of B, each including its respective cofactor sign. Since $\det(Bj)$ is simply a linear combination $\alpha_{1j}M_1 + \cdots + \alpha_{qj}M_q$, where α_{ij} is a $z \times z$ minor of $\oplus_{i=1}^{w_{d_2-d_1}} A$, there exists an ordering of M such that

$$\Lambda_z(\oplus_{i=1}^{w_{d_2-d_1}} A)M = J. \tag{4.2.1}$$

Now, since A is full rank, $\oplus_{i=1}^{w_{d_2-d_1}} A$ is full rank. Thus, by lemma 4.1, $\Lambda_z(\oplus_{i=1}^{w_{d_2-d_1}} A)$ is full rank. So, by 1.5, all entries of M are zero if and only if all entries of J are zero.

It remains to be shown that the entries of J are zeros or homogeneous $w_{d_2-d_1}$th degree polynomial functions, evaluated at P, from $k[\xi_1, \ldots, \xi_h]$.

To show this, consider X_j, a matrix consisting of X with z rows appended to the top such that $X_j(\oplus_{i=1}^{w_{d_2-d_1}} A) = B_j$ for $j = 1$ to t. X_j is possible to construct since we need only consider the z rows taken from $\oplus_{i=1}^{w_{d_2-d_1}} A$ that are appended to B to obtain B_j. If the row indices of the z rows of $\oplus_{i=1}^{w_{d_2-d_1}} A$ appended to B to obtain B_j are $r_{j_1}, \ldots, r_{j_z}$ respectively, we may append z rows to X such that the r_{j_i}th entry from the left on the ith row from the top is 1, and the rest along that ith row are zeros. This construction gives

$$X_j(\oplus_{i=1}^{w_{d_2-d_1}} A) = B_j, \tag{4.2.2}$$

. for j = 1, . . . , t. Thus, we simply must apply $\Lambda_{mw_{d_2-d_1}}$ to both sides of equation 4.2.2. Since B_j is $mw_{d_2-d_1} \times mw_{d_2-d_1}$, $\Lambda_{mw_{d_2-d_1}}(B_j) = \det(Bj)$. And since $\Lambda_{mw_{d_2-d_1}}$ is a functor, it respects function composition. Thus,

$$\Lambda_{mw_{d_2-d_1}}(X_j(\oplus_{i=1}^{w_{d_2-d_1}} A)) = \Lambda_{mw_{d_2-d_1}}(X_j)\Lambda_{mw_{d_2-d_1}}(X_j(\oplus_{i=1}^{w_{d_2-d_1}} A)) \tag{4.2.3}$$

Since X_j is $mw_{d_2-d_1} \times w_{d_1} \cdot w_{d_2-d_1}$ consisting of ones and zeros, $\Lambda_{mw_{d_2-d_1}}(X_j)$ is a row vector consisting of field elements. Since $\oplus_{i=1}^{w_{d_2-d_1}} A$ is $w_{d_1} \cdot w_{d_2-d_1} \times mw_{d_2-d_1}$, $\Lambda_{mw_{d_2-d_1}}(\oplus_{i=1}^{w_{d_2-d_1}} A)$ is a column vector consisting of entries dependent upon the entries of A.

Now, in the computation of the entries of $\Lambda_{mw_{d_2-d_1}}(\oplus_{i=1}^{w_{d_2-d_1}} A)$, we must select rows from $\oplus_{i=1}^{w_{d_2-d_1}} A$. Since $\oplus_{i=1}^{w_{d_2-d_1}} A$ is a matrix with $w_{d_2-d_1}$ copies of A along the diagonal with zeros in all other entries, and since $\text{rank}(A) = m$, if we select the $mw_{d_2-d_1}$ rows in such a way as to include more than m rows from any one copy of A, the determinant of that minor is zero. Otherwise, we must select exactly m rows from each of the $w_{d_2-d_1}$ copies of A. And the $mw_{d_2-d_1} \times mw_{d_2-d_1}$ minor obtained consists of $m \times m$ minors of A along the diagonal with all other entries equal to zero. Thus, the determinant of that particular minor is the product of those $w_{d_2-d_1} m \times m$ minors of A. Furthermore, the $mw_{d_2-d_1} \times mw_{d_2-d_1}$ minor of $\oplus_{i=1}^{w_{d_2-d_1}} A$ in question is a $w_{d_2-d_1}$th degree monomial function from $k[\xi_1, \ldots, \xi_h]$ evaluated at P.

Thus, the entries of $\Lambda_{mw_{d_2-d_1}}(\oplus_{1}^{w_{d_2-d_1}} A)$ are either zero or the product of $w_{d_2-d_1}$ not necessarily distinct Plücker coordinates.

Furthermore, $\det(B_j)$ is a linear combination of $w_{d_2-d_1}$th degree monomials comprised of Plücker coordinates for $j = 1, \ldots, t$.

Let $g_j(P) = \det(B_j)$, a homogeneous $w_{d_2-d_1}$th degree polynomial from the polynomial ring $k[\xi_1, \ldots, \xi_h]$ evaluated at P for $j = 1$ to t. The g_js are not necessarily distinct up to sign, as demonstrated in the proof of 3.4. Let s be the number of distinct $|gj|s$, so that by reordering, we have that $\{g_i\}$ for $i = 1$ to s consists of all distinct g_js up to sign.

Thus, all $w_{d_2} \times w_{d_2}$ minors of B are zero if and only if $g_i(P) = 0$ for all $i = 1, \ldots, s$. □

As we did with 3.4, we may restate 4.2 as follows:

4.3. **Corollary.** *Let $R = k[x_1, \ldots, x_n]$, m, d_1, d_2 be positive integers such that $m < w_{d_1}, d_1 < d_2, mw_{d_2-d_1} \geq w_{d_2}$, and I be an ideal of R generated by m linearly independent homogeneous d_1th degree polynomials. Furthermore, let $h = \begin{pmatrix} w_{d_1} \\ m \end{pmatrix}$, A be the coefficient matrix with respect to the monomial basis of the aforementioned m homogeneous d_1th degree polynomials, and $P = (P_1, \ldots, P_h) = \Lambda_m(A)^T$. Then there exists a projective variety V in $\boldsymbol{P}^{h-1}$ such that the vector space of homogeneous d_2th degree polynomials does not have image zero under $\varphi : R \to R/I$ if and only if $P \in V$.*

5. CONCLUDING REMARKS.

In this paper, we explored a relationship between finite dimensional commutative algebras (i.e. quotients of polynomial rings) and projective varieties. In the proof of theorem 4.2, we showed that if I is an ideal of the polynomial ring $k[x_1, \ldots, x_n]$ generated by m linearly independent d_1th degree polynomials with the coefficient matrix of these polynomials given by A, then there exists a projective variety V such that the vector space of homogeneous d_2th degree polynomials has image zero under $\varphi : R \to R/I$ if and only if $\Lambda_m(A)^T \notin V$.

Through the proof of 4.2, we showed that to obtain the functions $g_1, \cdots, g_s$ defining the projective variety V mentioned in corollary 4.3, we must calculate the determinants of the B_js, constructed by appending rows from $\oplus_{i=1}^{w_{d_2-d_1}} A$ to B. We must then determine whether the point P is on the projective variety V.

Though 4.2 assures us of the existence of the functions $g_1, \ldots, g_s$, it does not offer any ease in the laborious computation necessary in obtaining such functions. Consider the following challenge [4]:

Let $R = k[x_1, \ldots, x_5]$, I be an ideal generated by ten linearly independent quadratics. We wish to assure that the vector space of homogeneous quartics has image zero under $\varphi : R \to R/I$.

Even in this seemingly innocuous problem, a solution is virtually unattainable- the sheer number of Plücker coordinates in the coefficient matrix A, 3003, presents a daunting challenge in computation. Upon obtaining them, we must obtain B, a 35×50 matrix. We must then attach 15 rows taken from $\oplus_{i=1}^{5} A$ to the top of B to obtain the B_js. Notice, we obtain $\begin{pmatrix} 75 \\ 15 \end{pmatrix} \approx 2.28 \times 10^{15}$ different B_js. Thus, we must evaluate 2.28×10^{15} different determinants to obtain the $g_1, \ldots, g_s$, a daunting

[4]This challenge is motivated by the Gorenstein ring providing the counterexample to one of Auslander's conjectures discussed in Jorgensen and Sega.

challenge in computation. Only with the advantage provided by computer algebra software will these results be of practical benefit in cases more complicated than those analyzed herein.

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